# METRICS ON IDEALS AND THE HAUSDORFF DISTANCE

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## INTRODUCTION

For approximately finite-dimensional algebras, Aguilar introduced a metric that calculates distances between ideals of the algebra. When the algebra is commutative, it is isomorphic to a space of complex-valued continuous functions on a compact metric space, $(X, d)$. The ideals in this case are in bijection with the closed subsets of $X$, which induces a metric on the closed subsets of $X$, called the ideal metric. However, there is a well-known metric on the closed subsets of any compact metric spaces called the Hausdorff distance.

Therefore, our work focuses on comparing the Hausdorff distance with this ideal metric. For a particular $X$, we have calculated the Hausdorff distance and ideal metric on certain subsets and have discovered cases where they agree or disagree. These comparisons suggest that the ideal metric is not ideal, and current/future work will develop a new ideal metric that behaves similarly to the Hausdorff distance.

## HAUSDORFF DISTANCE

Given a metric space $(X, d)$, $\text{Haus}_{d_1}$ is a metric on compact subsets of $X$. For instance, for $(\mathbb{R}, d_1)$, we have

$$\text{Haus}_{d_1}([0,1],[1/2,1]) = 1/2$$

$$\text{Haus}_{d_1}([0,1],[1/2,2]) = 1$$

## IDEAL SPACE

Consider the set

$$A = \left\{ \frac{1}{2^{n-1}} : n \in \mathbb{N} \right\} \cup \{0\},$$

and the set of complex valued continuous functions on $A$,

$$C(A) = \{ f : A \to \mathbb{C} : f \text{ is continuous} \}.$$

Given $B \subseteq A$, closed, the subring

$$I_B = \{ f \in C(A) : f(x) = 0 \; \forall x \in B \}$$

is an ideal of $C(A)$. The ideal space, then, is denoted by

$$\text{Ideal}(C(A)) = \{ I_B \subseteq C(A) : B \subseteq A \text{ is closed} \}.$$

Note, $B \subseteq A$, closed $\mapsto I_B \in \text{Ideal}(C(A))$ is injective.

## METRICS ON IDEALS

For each $n \in \mathbb{N}$, define the eventually constant functions at $\frac{1}{2^{n-1}}$

$$C_n = \left\{ f \in C(A) : f(x) = f\left(\frac{1}{2^{n-1}}\right) \; \forall x \leq \frac{1}{2^{n-1}} \right\}$$

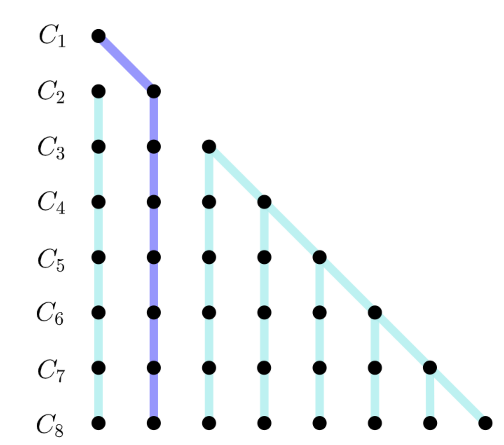
Given ideals $I_B$ and $I_C$, define a metric by

$$d_\varphi(I_B, I_C) = \begin{cases} 0 & I_B = I_C \\ 2^{-\min\{m \in \mathbb{N} : I_B \cap C_m \neq I_C \cap C_m\}} & I_B \neq I_C \end{cases},$$

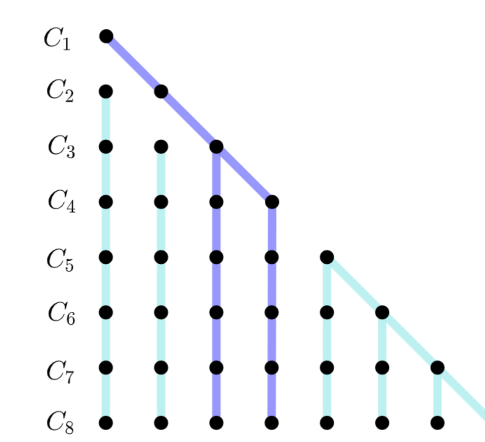
and define

$$d_{\varphi_I}(B, C) = d_\varphi(I_B, I_C)$$

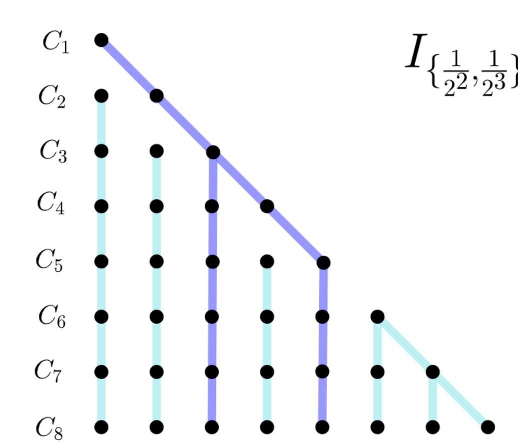
The ideals can be represented by graphs called Bratteli diagrams, which assist in calculating distances in this metric:



$I_{\{\frac{1}{2}\}}$



$I_{\{\frac{1}{2^2},\frac{1}{2^3}\}}$



$I_{\{\frac{1}{2^2},\frac{1}{2^4}\}}$

## CALCULATIONS IN $\text{Haus}_{d_1}$ AND $d_{\varphi_I}$

Thus, $\text{Haus}_{d_1}$ and $d_{\varphi_I}$ are both metrics on the compact subsets of $A$. Let $m, n, k \in \mathbb{N}$. Then

$$\text{Haus}_{d_1}\left( \left\{ \frac{1}{2^m} \right\}, \left\{ \frac{1}{2^n}, \frac{1}{2^{n+k}} \right\} \right) = \begin{cases} \left| \frac{1}{2^m} - \frac{1}{2^{n+k}} \right|, & m \leq n \\ \left| \frac{1}{2^m} - \frac{1}{2^n} \right|, & m > n \end{cases}$$

$$d_{\varphi_I}\left( \left\{ \frac{1}{2^m} \right\}, \left\{ \frac{1}{2^n}, \frac{1}{2^{n+k}} \right\} \right) = 2^{-(\min\{m,n\}+2)}$$

For example,

$$\text{Haus}_{d_1}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^3} \right\} \right) = \frac{1}{2} - \frac{1}{8}$$

$$\text{Haus}_{d_1}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^4} \right\} \right) = \frac{1}{2} - \frac{1}{16}$$

and,

$$d_{\varphi_I}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^3} \right\} \right) = \frac{1}{8} = d_{\varphi_I}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^4} \right\} \right)$$

## OBSERVATIONS & FUTURE WORK

Our calculations tell us that even though the Hausdorff distance and $d_{\varphi_I}$ induce the same topology, the metrics behave quite differently. Indeed,

$$\text{Haus}_{d_1}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^3} \right\} \right) \neq \text{Haus}_{d_1}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^4} \right\} \right)$$

However,

$$d_{\varphi_I}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^3} \right\} \right) = d_{\varphi_I}\left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{2^2}, \frac{1}{2^4} \right\} \right)$$

This observation suggests that the rate of convergence of particular sequences of ideals may be quite between the metrics. Thus, we hope to create a metric that behaves more like the Hausdorff distance by using the ideal/algebraic structure of $C(A)$. Our idea is to capture more properties of the ideals by using the Bratteli diagram.

## References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