Coloring Abelian Cayley Graphs

Jonathan Cervantes

Dr. Mike Krebs

CAL STATE LA
California State University, Los Angeles

What is a Cayley graph?

Let G be a group and let $S \subseteq G$. The Cayley graph of G with respect to S, denoted $\mathrm{Cay}(G, S)$, is the graph with vertex set G and edge set $\{(g,\ g+s) \ :\ g \in G,\ s \in S\}$.

We consider Cayley graphs $\mathrm{Cay}(G, S)$ where G is abelian and S is a finite symmetric subset of G that generates G.

Proper Coloring. A *proper coloring* of a graph G is a mapping from the vertex set to a set of "colors" so that adjacent vertices are mapped to different elements of the set.

Setup

Take $\mathrm{Cay}(G, S)$ where $S = \{\pm s_1, \pm s_2, \ldots, \pm s_n\}$ generates G. Take $\varphi : \mathbb{Z}^n \to G$ given by $e_i \mapsto s_i$ where e_i is the element with 1 in the ith coordinate and 0 elsewhere. Let $K = \ker \varphi$. Then φ induces a graph isomorphism between $X = \mathrm{Cay}(\mathbb{Z}^n/K, \{K \pm e_1, K \pm e_2, \ldots, K \pm e_n\})$ and $\mathrm{Cay}(G, S)$.

Theorem

Let $x \in \mathbb{Z}^n \setminus \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ and $K = \langle x \rangle$. Then

$$\chi(X) = \begin{cases} 2 & \text{if } \sum_{i=1}^{n} x_i \text{ is even} \\ 3 & \text{otherwise} \end{cases}$$

If $x \in \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$, then X has loops and cannot be properly colored.

Proof

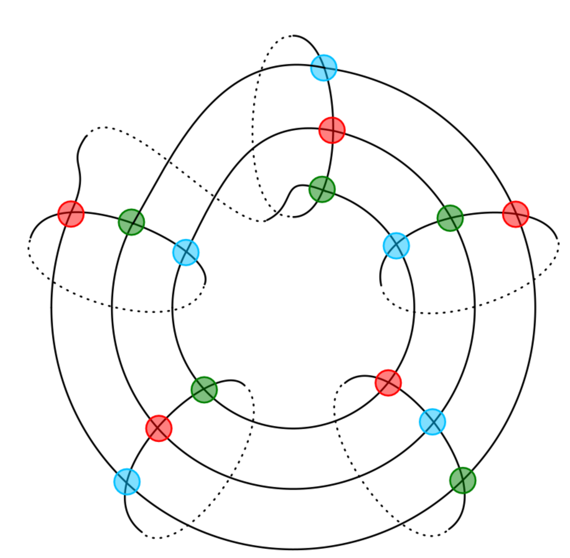
Let $s = \sum_{i=1}^{n} |x_i|$. Take $\psi : \mathbb{Z}^n/K \to \mathbb{Z}/\langle s \rangle$ where $e_i \mapsto -1$ if $x_i < 0$ and $e_i \mapsto 1$ otherwise. This gives a graph homomorphism from X to the s-cycle $\mathrm{Cay}(\mathbb{Z}/\langle s \rangle, \{\pm 1\})$ and thus gives an upper bound for $\chi(X)$. Note that X contains an odd cycle whenever s is odd (trace a cycle from the origin to x).

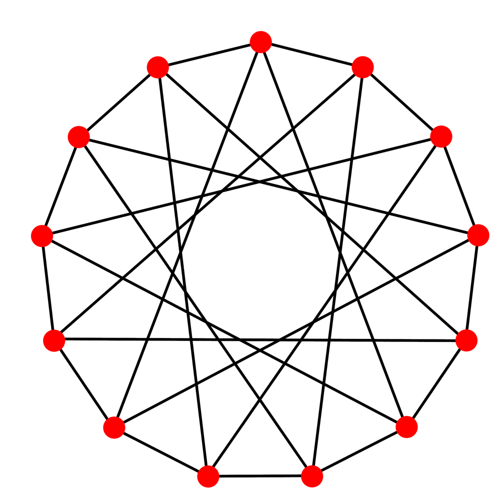
Matrix Form

We wish to encode X in a matrix when $K = \langle x_1, x_2, \ldots, x_k \rangle$ where $x_1, x_2, \ldots, x_k \in \mathbb{Z}^n$. We obtain an $n \times k$ matrix M_X by letting x_i be the ith column. We call M_X an *associated matrix* of X and refer to a graph X of this form as a *standardized abelian Cayley graph*, which we abbreviate SACG. We give M_X a superscript SACG to denote that the matrix represents such a graph. We sometimes give the matrix a subscript X to denote the associated graph (*e.g.* $\left(\begin{smallmatrix} 1 & 0 \\ -5 & 15 \end{smallmatrix}\right)^{\mathrm{SACG}}_X$). Note that M_X is not unique.

Example

The circulant graph $\mathrm{Cay}(\mathbb{Z}_{15}, \{\pm 1, \pm 5\})$ is isomorphic to $\left(\begin{smallmatrix} 1 & 0 \\ -5 & 15 \end{smallmatrix}\right)^{\mathrm{SACG}}$ pictured below (left). Similarly, $\left(\begin{smallmatrix} 1 & 0 \\ 5 & 13 \end{smallmatrix}\right)^{\mathrm{SACG}}$ is isomorphic to the circulant graph $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm 1, \pm 5\})$ pictured below (right).





Lemma

Let X be a standardized abelian Cayley graph with an associated $m \times n$ matrix M_X.

1. We obtain a graph homomorphism by reducing a column by a common factor.
2. Let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ be a surjective function. We obtain a graph homomorphism from the mapping $e_i \mapsto e_{f(i)}$ where $e_i \in \mathbb{Z}^m$ and $e_{f(i)} \in \mathbb{Z}^k$.
3. A graph isomorphism is obtained from the map given by $e_j \mapsto -e_j$ and $e_i \mapsto e_i$ for $i \neq j$ where $e_i, e_j \in \mathbb{Z}^m$.

Lemma

Let X and X' be standardized abelian Cayley graphs with associated matrices M_X and $M_{X'}$, respectively.

1. If $M_{X'}$ is obtained by permuting the columns of M_X, then $X = X'$.
2. If $M_{X'}$ is obtained by multiplying a column of M_X by -1, then $X = X'$.
3. Suppose x_j and x_i are the jth and ith columns of M_X, respectively, with $j \neq i$. If $M_{X'}$ is obtained by replacing the jth column of M_X with $x_j + a x_i$ for some integer a, then $X = X'$.
4. If $M_{X'}$ is obtained by deleting any column from M_X which is in the $\mathbb{Z}$-span of the other columns, then $X = X'$.
5. If $M_{X'}$ is obtained by permuting the rows of M_X, then X is isomorphic to X'.
6. If $M_{X'}$ is obtained by multiplying a row of M_X by -1, then X is isomorphic to X'.

Let A be a matrix. By performing row and column operations as above, one can show that A^{SACG} has a lower triangular associated matrix.

Theorem

Let X be a standardized abelian Cayley graph with an associated matrix $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ where $a \geq 0$ and $c \geq 0$. Let $d = \gcd(a, b)$ and $e = \gcd(a, b, c)$. Then:

1. If either *(i)* $c = 1$ or *(ii)* $a = 1$ and $c \mid b$ or *(iii)* $a = 0$ and $\gcd(b, c) = 1$, then X has loops and is not properly colorable.
2. If both $a + b$ and c are even, then $\chi(X) = 2$.
3. If *(i)* neither of the conditions in the previous statements hold, and *(ii)* $a = 0$ or $e > 1$ or $c \mid b$, then $\chi(X) = 3$.
4. If none of the conditions of the previous statements hold, let q be the product of all primes p such that $p \mid a$ but $p \nmid d$. (If there are no such primes, $q = 1$.) Then
$$\chi(X) = \chi(\mathrm{Cay}(\mathbb{Z}_{ac}), \{\pm a, \pm(b + qc)\}).$$

Results

Let A be a 2×2 matrix. Suppose A^{SACG} does not contain loops. If $3 \mid \det A$, then $\chi(A^{\mathrm{SACG}}) \leq 3$.

Let A be an $m \times n$ matrix and let a_{ij} denote its entry in the ith row and jth column. Then $\chi(A^{\mathrm{SACG}}) = 2$ if and only if $\sum_{i=1}^{n} a_{ij}$ is even for each j.

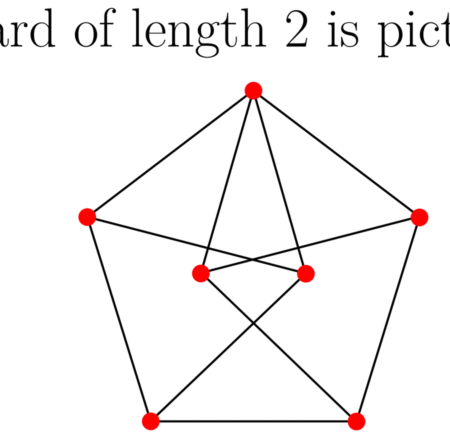
We suspect that the following unproven claims will summarize our results.

Claim. Let A be a 3×2 matrix. Suppose that A contains no zero rows and that A^{SACG} does not contain loops. Then A^{SACG} is 3-colorable unless it has one of the following as an associated matrix:

$$\begin{pmatrix} 1 & 0 \\ -1 & a \\ -1 & a + 3(k-1) \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3k & 1 + 3k \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3\ell & 2 \end{pmatrix}$$

where $a \in \mathbb{Z}$ with $3 \nmid a$, $k \in \mathbb{Z}^+$, and $\ell \in \mathbb{Z}$.

Diamond Lanyard. An *unclasped diamond lanyard of length* 1 is a diamond. The *endpoints* of an unclasped diamond lanyard are its two degree 2 vertices. Recursively, we define an *unclasped diamond lanyard U of length $\ell + 1$* to be the union of an unclasped diamond lanyard Y of length ℓ and a diamond D, such that the intersection of Y and D is a common endpoint of Y and D. A *(clasped) diamond lanyard of length ℓ* is obtained by adding to an unclasped diamond lanyard U of length ℓ an edge between the endpoints of U. A diamond lanyard of length 2 is pictured below.



Claim. Let A be a matrix with at most 3 rows and at most 2 columns. Let $X = A^{\mathrm{SACG}}$ and suppose X has no loops. Then $\chi(X) \leq 4$ if and only if X does not contain a 5-clique and $\chi(X) \leq 3$ if and only if X does not contain a 5-clique nor a diamond lanyard nor a $C_{13}(1, 5)$.