Crux 48(2), Problem 4716

The three roots of the cubic $x^3 + 4x^2 + 4x + 1 = 0$ are the slopes of the sides of a triangle. Find the slope of its Euler line.

Crux 48(2), Problem 4716, Generalized

Clayton Coe, Cal Poly Pomona

The three roots of the cubic $x^3 + bx^2 + bx + 1 = 0$ are the slopes of the sides of a triangle. Find the slope of its Euler line.

Original Problem: $b = 4$

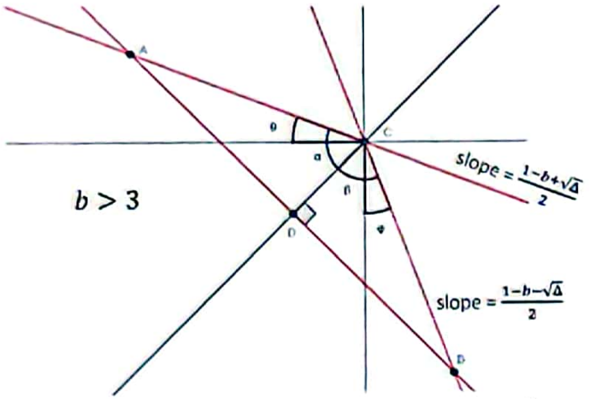
$$x^3 + bx^2 + bx + 1$$

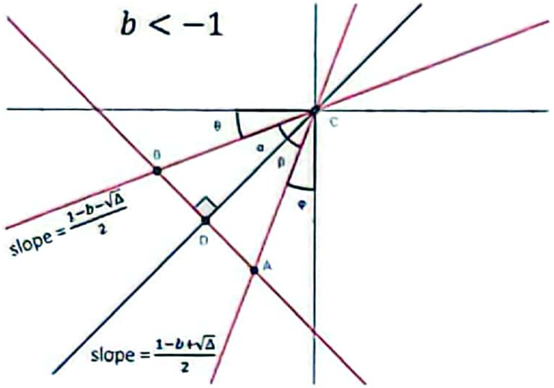
$$= (x+1)(x^2 + (b-1)x + 1)$$

The roots are $x = -1$ and $\frac{(1-b)\pm\sqrt{(b-1)^2-4}}{2}$

The roots are the values of the slopes of the triangle.

Are these isosceles?





$$\Delta = (b-1)^2 - 4$$

$$\tan(180^\circ - \theta) = -\tan\theta = \frac{1 - b + \sqrt{\Delta}}{2}$$

$$\tan\theta = \frac{b - 1 - \sqrt{\Delta}}{2}$$

$$\tan(90^\circ + \phi) = -\cot\phi = \frac{-1}{\tan\phi} = \frac{1 - b - \sqrt{\Delta}}{2}$$

$$\tan\phi = \frac{2}{b - 1 - \sqrt{\Delta}} \cdot \frac{b - 1 + \sqrt{\Delta}}{b - 1 + \sqrt{\Delta}}$$

$$\tan\phi = \frac{2(b - 1 + \sqrt{\Delta})}{(b-1)^2 - (b-1)^2 + 4} = \frac{b - 1 + \sqrt{\Delta}}{2}$$

$$\tan\theta = \frac{1 - b - \sqrt{\Delta}}{2}$$

$$\tan(90^\circ - \phi) = \cot\phi = \frac{1}{\tan\phi} = \frac{1 - b + \sqrt{\Delta}}{2}$$

$$\tan\phi = \frac{2}{1 - b + \sqrt{\Delta}} \cdot \frac{1 - b - \sqrt{\Delta}}{1 - b - \sqrt{\Delta}}$$

$$\tan\phi = \frac{2(1 - b - \sqrt{\Delta})}{(1-b)^2 - (b-1)^2 + 4} = \frac{1 - b - \sqrt{\Delta}}{2}$$

$$\tan\theta = \tan\varphi \Rightarrow \theta = \varphi \Rightarrow \alpha = \beta$$

$\overline{CD}$ is the altitude and angle bisector of $\overline{AB}$.

Altitude + Angle Bisector = Isosceles!

The Euler Line of an isosceles triangle is perpendicular to its base

$\overline{CD}$ is the Euler line, with a slope of 1.