Inverted Cassini Ovals and Surfaces

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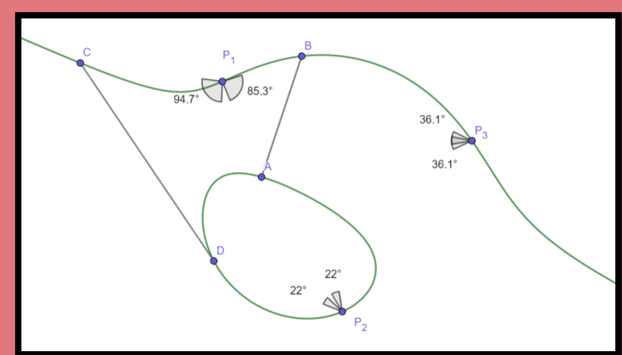
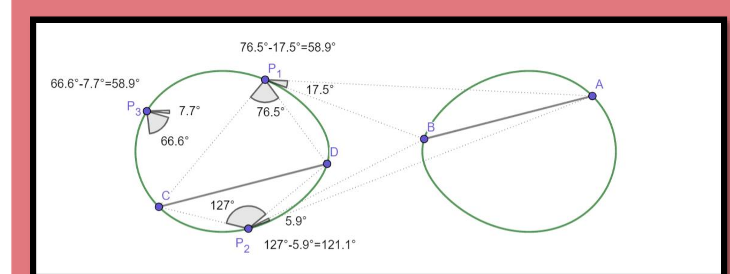
Introduction

An Apollonian cubic (as shown on the right) is part of the locus of points which see two segments at equal or supplementary angles[3].





A Cassini Oval is a collection of points for which the products of two distances from given foci to these points is constant

It has been shown that two chords that form a diameter of a Cassini oval subtend angles whose difference is constant at any point of the oval (See above figure)[1].

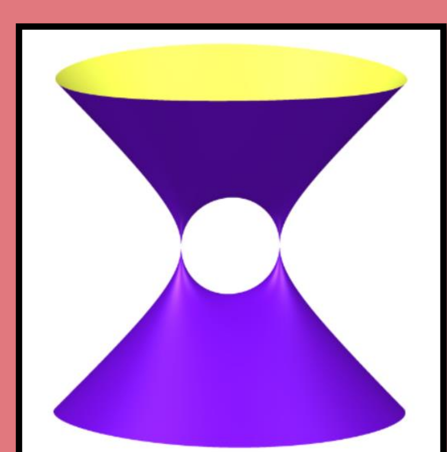
The angle differences, sums, and equalities above can be generalized to oriented angle sums whose values are between $-\pi$ and π .

Define the Oriented Dissected θ -Isoptic (θ -Disoptic) to be the points which view two closed curves at oriented angles which sum to θ or $\theta - \pi$ for some $\theta \in (0, \pi]$.

From the above it follows that the Apollonian cubic is an oriented π -disoptic for two segments and a Cassini oval is an oriented θ -disoptic for congruent collinear segments.

A Cassini Surface is a surface whose level curves are Cassini ovals, all with the same foci.
Given by the following equation:

$$((x-a)^2 + y^2)((x+a)^2 + y^2) - z^4 = 0$$



Background and Notation

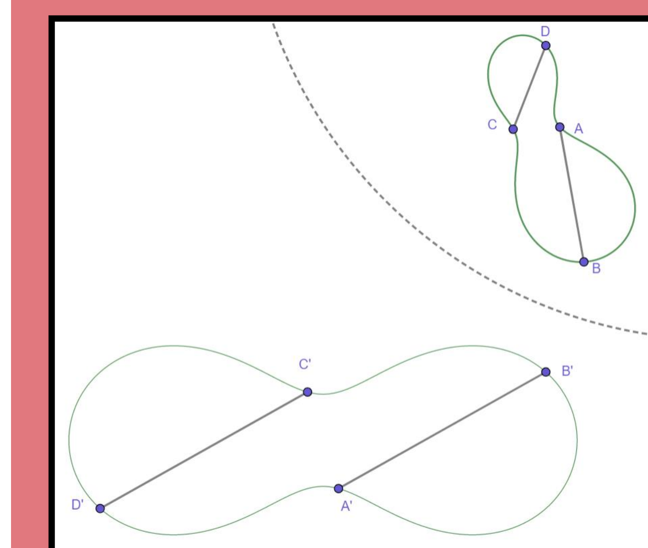
-Let $\measuredangle(APB)$=$\measuredangle APB$ if the path from A to P to B is clockwise.
-Many of the results on the curves are derived from the principal value of a complex number and the Mobius Transform.
-The circle inversion of a point P over some circle with center O and radius r is the point P' on $\overrightarrow{OP}$ such that the product of OP and OP' is r^2.

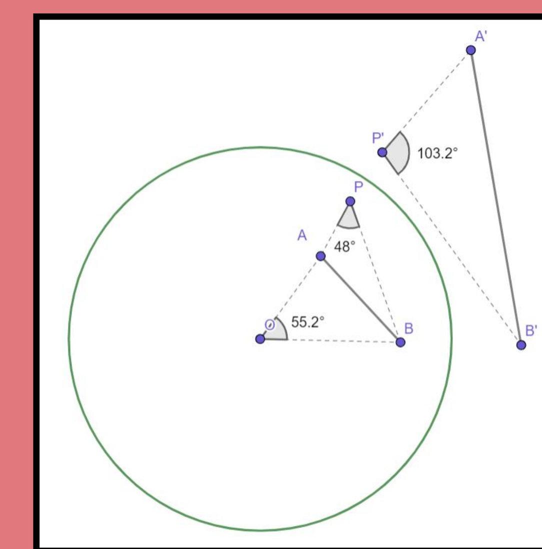
Results: Curves

Theorem: Given segments AB and CD such that ABCD is a parallelogram the θ -disoptic of AB and CD is a Cassini oval, a right hyperbola, or two perpendicular lines.

Lemma: For unique points A,P, B, and O and their respective inversions A',P', and B' over some circle with center O we have

$$\measuredangle(B'P'A') = \measuredangle(BOA) + \measuredangle(APB) \bmod(-\pi, \pi]$$





Lemma: For any four points A,B, C, and D there is some circle which inverts the points, so they form a parallelogram ABCD, labeled in that orientation.

Theorem: The oriented θ -disoptic of two segments is the circle inversion of a Cassini oval or two perpendicular lines.

Results: Surfaces

It can be derived for vectors $u = \overrightarrow{PA}$ and $v = \overrightarrow{PB}$ that $\sin(\measuredangle(APB)) = \frac{|u \times v|}{\|u\|\|v\|}$ and $\cos(\measuredangle(APB)) = \frac{(u \cdot v)}{\|u\|\|v\|}$

Hence, for vectors $u = \overrightarrow{PA}$, $v = \overrightarrow{PB}$, $r = \overrightarrow{PC}$, and $s = \overrightarrow{PD}$ we have

$$\tan(\measuredangle(APB) + \measuredangle(CPD)) = \frac{\sin(\measuredangle(APB)+\measuredangle(CPD))}{\cos(\measuredangle(APB)+\measuredangle(CPD))} = \frac{(|u\times v|)(r\cdot s)+(u\cdot v)(|r\times s|)}{(u\cdot v)(r\cdot s)-(|u\times v|)(|r\times s|)}$$

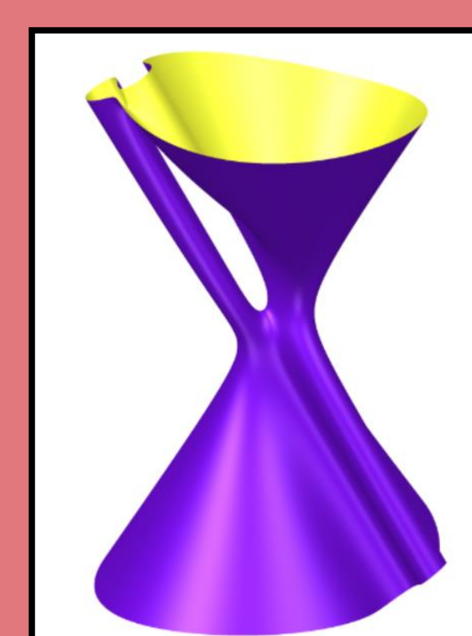
And so, the equation for an oriented θ-disoptic, for $\theta \in (0, \pi)$, of two segments is given by

$$\cot(\theta)[(|u \times v|)(r \cdot s) + (u \cdot v)(|r \times s|)] - [(u \cdot v)(r \cdot s) - (|u \times v|)(|r \times s|)] = 0$$

Letting $z = \cot(\theta)$, we have a surface given by a polynomial.

This surface is akin to a Cassini Surface since its horizontal cross sections are inverted Cassini ovals.

Theorem: In general, these surfaces are irreducible quartics with no singularities. They become reducible with singularities if any of the segment endpoints are the same.



Discussion and Further Research

- We arrived at a classification of oriented θ –disoptics for two segments.
  - For three segments, in general there is no circle inversion analog for the second lemma. So a classification for n segments will have more interesting curves.
  - Other research exists on equioptics[2] (similar to π – disoptics) of two closed curves. Our methods here become cumbersome when considering any closed curves. This is because the points of tangency are moving.
- Apollonian cubics have a natural group structure using the chord and tangent addition rule. We are interested to see if there is some similar group structure on inverted Cassini ovals.
- In essence this research extends the inscribed angle theorem to two segments.
  - In this vein, the angle for the oriented disoptic can be determined via the foci and center of inversion of the inverted Cassini oval.
- These new surfaces were meant to extend the notion of a Cassini Surface. While they do this, we are still learning about them.
  - Cassini Surfaces have a vertical cross section of a circle and two hyperbola. We hope to find interesting curves that lie on these 'inverted Cassini Surfaces' in future work.

Citations

[1] Mathematical Questions and Solutions in Continuation of the Mathematical Columns of "the Educational Times", Vol. 51, pg 41, F. Hodgson, 1889, Digitized Nov. 18 2009
[2] B. Odenhal, Equioptic Curves of Conic Sections, *Journal for Geometry and Graphics*, Vol 14 (2010), No. 1, 29-43.
[3] P. Pamfilos and A. Thoma, Apollonain Cubics: An Application of Group Theory to a Problem in Euclidean Geometry, *Mathematics Magazine*, Vol. 72, No. 5 (Dec. 1999), pp. 356-366, Taylor&Francis, https://www.jstor.org/stable/2690791

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